

A Rodrigues formula for the Jack polynomials and the Macdonald-Stanley conjecture

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Abstract

A formula of Rodrigues-type for the Jack polynomials is presented. It is seen to imply a weak form of a conjecture of Macdonald and Stanley.

1 Introduction

As usual, a partition λ of N will be a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers in decreasing order $\lambda_1 \geq \lambda_2 \geq \dots$ such that $|\lambda| = \lambda_1 + \lambda_2 + \dots = N$. The integers λ_i are called the parts of λ . Let λ and μ be two partitions of N . In the dominance ordering, $\lambda \geq \mu$ if $\lambda_1 + \lambda_2 + \dots + \lambda_i \geq \mu_1 + \mu_2 + \dots + \mu_i$ for all i .

Let Λ_n denote the ring of symmetric functions in the variables x_1, x_2, \dots, x_n . Two natural bases of Λ_n are: the power sums $p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots$ where $p_i = \sum_k x_k^i$ and the symmetric monomials $m_\lambda = \sum x_1^{\lambda_1} x_2^{\lambda_2} \dots$ summed over distinct permutations.

To the partition λ with $m_i(\lambda)$ parts equal to i , we associate the number

$$z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \dots \quad (1)$$

Let α be a parameter and $\mathbb{Q}(\alpha)$ the field of all rational functions of α with rational coefficients. A scalar product $\langle \cdot, \cdot \rangle$ on $\Lambda_n \otimes \mathbb{Q}(\alpha)$ is defined by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \alpha^{\ell(\lambda)}, \quad (2)$$

where $\ell(\lambda)$ is the number of parts of λ . The Jack polynomials $J_\lambda(x_1, \dots, x_n; \alpha) \in \Lambda_n \otimes \mathbb{Q}(\alpha)$ are uniquely specified by [1, 2]

$$(i) \quad \langle J_\lambda, J_\mu \rangle = 0, \quad \text{if } \lambda \neq \mu, \quad (3)$$

$$(ii) \quad J_\lambda = \sum_{\mu \leq \lambda} v_{\lambda\mu}(\alpha) m_\mu, \quad (4)$$

$$(iii) \quad \text{if } |\lambda| = N, \quad v_{\lambda, 1^N} = N! \quad (5)$$

The Jack polynomials $J_\lambda(x_1, \dots, x_n; \alpha)$ where $\ell(\lambda) \leq n$ are also known to obey the differential equations [3, 1]

$$H(\alpha) J_\lambda(x; \alpha) = \varepsilon_\lambda(\alpha) J_\lambda(x; \alpha), \quad (6)$$

where

$$H(\alpha) = \alpha \sum_{j=1}^n \left(x_j \frac{\partial}{\partial x_j} \right)^2 + \sum_{j < k} \left(\frac{x_j + x_k}{x_j - x_k} \right) \left(x_j \frac{\partial}{\partial x_j} - x_k \frac{\partial}{\partial x_k} \right), \quad (7)$$

and

$$\varepsilon_\lambda(\alpha) = \sum_{j=1}^n [\alpha \lambda_j^2 + (n + 1 - 2j) \lambda_j]. \quad (8)$$

The polynomials J_λ appear in the wave functions of the Calogero-Sutherland model [4]. This is an exactly solvable quantum-mechanical system which describes n particles on a circle interacting pairwise through long-range potentials. It is nowadays

quite useful in studies of phenomena associated with fractional statistics and there is considerable interest in identifying the algebraic structure underlying this model. In this connection, it is natural to look for a formula giving the wave functions of the excited states through the action of creation operators on the ground state wave function. We obtained [3] as a result a formula of Rodrigues-type for the Jack polynomials which implies a weak form of a longstanding conjecture due to Macdonald and Stanley, namely that the coefficients $v_{\lambda\mu}(\alpha)$ in (4) are polynomials in α with integer coefficients. We present and discuss this formula here. It will be proved elsewhere [3].

2 Creation operators

The creation operators are constructed from the Dunkl operators [5]

$$D_i = \alpha x_i \frac{\partial}{\partial x_i} + \sum_{\substack{j=1 \\ j \neq i}}^n \frac{x_i}{x_i - x_j} (1 - K_{ij}), \quad i = 1, 2, \dots, n, \quad (9)$$

where $K_{ij} = K_{ji}$ is the operator that permutes the variables x_i and x_j :

$$K_{ij}x_i = x_jK_{ij}, \quad K_{ij}D_i = D_jK_{ij}, \quad K_{ij}^2 = 1. \quad (10)$$

Let $J = \{j_1, j_2, \dots, j_\ell\}$ be sets of cardinality $|J| = \ell$ made of integers $j_\kappa \in \{1, \dots, n\}$, $1 \leq \kappa \leq \ell$ such that $j_1 < j_2 < \dots < j_\ell$ and introduce the operators

$$D_J = (D_{j_1} + 1)(D_{j_2} + 2) \cdots (D_{j_\ell} + \ell), \quad (11)$$

labelled by such sets. The creation operators B_i^+ are defined by

$$B_i^+ = \sum_{\substack{J \subset \{1, \dots, n\} \\ |J|=i}} x_J D_J, \quad (12)$$

with

$$x_J = \prod_{i \in J} x_i. \quad (13)$$

The sum in (12) is over all subsets J of $\{1, \dots, n\}$ that are of cardinality i .

3 The Rodrigues formula

We can now state our main result [3].

Theorem 1. *The Jack polynomials $J_\lambda(x; \alpha)$ associated to partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ are given by*

$$J_\lambda(x; \alpha) = (B_n^+)^{\lambda_n} (B_{n-1}^+)^{\lambda_{n-1} - \lambda_n} \dots (B_1^+)^{\lambda_1 - \lambda_2} \cdot 1. \quad (14)$$

That the right-hand side of (14) is a symmetric polynomial of degree $N = \lambda_1 + \dots + \lambda_n$ in the variables x_1, \dots, x_n is easily seen from the properties of the Dunkl operators. It is checked that the operators (9) satisfy the commutation relations

$$[D_i, D_j] = (D_j - D_i)K_{ij}. \quad (15)$$

Let us now use the notation $\text{Res}^{\{i,j,k,\dots\}} X$ or $\text{Res}^J X$ to indicate that the operator X is taken to act on functions that are symmetric in the variables x_i, x_j, x_k, \dots or $x_{j_1}, x_{j_2}, \dots, j_\kappa \in J$, respectively. With m some integer, it is straightforward to verify with the help of (15) that

$$\text{Res}^{\{i,j\}}(D_i + m)(D_j + m + 1) = \text{Res}^{\{i,j\}}(D_j + m)(D_i + m + 1). \quad (16)$$

It follows that $\text{Res}^J D_J$ is invariant under the permutations of the variables $x_{j_\kappa}, j_\kappa \in J$ and that this operator therefore leaves invariant the space of symmetric functions in these variables.

Recalling how the creation operators B_i^+ are constructed in terms of the operators D_J , it is then clear that $(B_n^+)^{\lambda_n} \dots (B_1^+)^{\lambda_1 - \lambda_2} \cdot 1$ is a symmetric function of the variables x_1, \dots, x_n . That it is a homogeneous polynomial of degree N is readily seen from observing that B_i^+ has scaling dimension i .

Let $\varphi_{(\lambda_1, \dots, \lambda_i, 0, \dots)} = (B_i^+)^{\lambda_i} \dots (B_1^+)^{\lambda_1 - \lambda_2} \cdot 1$. The proof of the theorem involves showing that

$$[H(\alpha), B_i^+] \varphi_{(\lambda_1, \dots, \lambda_i, 0, \dots)} = B_i^+ \left\{ 2\alpha \sum_{j=1}^n \lambda_j + i\alpha + i(n - i) \right\} \varphi_{(\lambda_1, \dots, \lambda_i, 0, \dots)}. \quad (17)$$

If (17) is true, it is clear that successive applications of the creation operators on 1 will build eigenfunctions of H . One then iteratively checks that the spectrum coincides with (8) to confirm the identification (14). A lengthy combinatorial argument which will be published elsewhere [3] actually leads to (17).

4 The conjecture of Macdonald and Stanley

Many conjectures involving the Jack polynomials have been formulated. A famous one is due to Macdonald and is reproduced in Stanley's reference article.[1] It is stated as follows.

Conjecture. Let

$$\tilde{v}_{\lambda\mu}(\alpha) = \frac{v_{\lambda\mu}(\alpha)}{\prod_{i \geq 1} m_i(\mu)!}, \quad (18)$$

where $v_{\lambda\mu}(\alpha)$ are as in (4). Then $\tilde{v}_{\lambda\mu}(\alpha)$ are polynomials in α with nonnegative integer coefficients.

To our knowledge, it was still not even known whether $v_{\lambda\mu}(\alpha)$ are polynomials. This can now be proved and the following weak form of the above conjecture is seen to follow directly from formula (14).

Theorem 2. *The coefficients $v_{\lambda\mu}(\alpha)$ in (4) are polynomials in α with integer coefficients.*

The proof proceeds in a recursive fashion. Assume that the assertion is true for partitions $\lambda = (\lambda_1, \dots, \lambda_i, 0, \dots)$. Formula (14) allows to increase the values of the parts and their number by acting on $J_\lambda(x; \alpha)$ with B_k^+ , $k \geq i$. Theorem 2 will be proved in these higher cases and thus in general since one can start with $J_0(x; \alpha) = 1$, if the matrix elements of the creation operators in the symmetric monomial basis are shown to be polynomials in α with integer coefficients. To convince oneself of that, let $m_\mu = \sum_{\text{distinct perm}} \hat{m}_\mu$ with $\hat{m}_\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$ be one such symmetric monomial. We know that $B_k^+ m_\mu = \sum_{\text{distinct perm}} B_k^+ \hat{m}_\mu$ is a symmetric function. Moreover, the operators D_m , $m = 1, \dots, n$, and hence the operators B_k^+ are easily found to give when acting on monomials \hat{m}_μ , sums of similar monomials multiplied

by polynomials in α with integer coefficients. Owing to the symmetry of $B_k^+ m_\mu$, $\sum_{\text{distinct perm}} B_k^+ \hat{m}_\mu$ must produce the desired result for the matrix elements of the creation operators in the symmetric monomial basis.

5 Conclusion

We believe that the Rodrigues formula that we have obtained will provide a useful tool to further advance the proofs of the outstanding conjectures on the Jack polynomials. We also trust that it will help obtain the dynamical algebra of the Calogero-Sutherland model. We are currently developing the extensions of the results presented here to the Macdonald polynomials [2] as well as to the multivariate special functions associated to lattices other than the one associated to A_{n-1} . We hope to report on these issues in the near future.

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